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Non-stationary finite amplitude convection

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There are stationary solutions of finite amplitude convection in a layer of fluid heated from below which show increasing heat transport with decreasing Rayleigh number in the neighbourhood of its critical value. It is shown that those solutions are unstable and that convection with periodic time dependence can occur in these cases, when the heat flux is the given parameter instead of the temperature difference between the boundaries of the layer. The time dependence has been calculated explicitly for the case of convection with temperature variation of the material properties.

1. Introduction

The problem of finite amplitude convection in a layer heated from below usually is treated with the Rayleigh number as given. This parameter occurs in the governing equations of the problem as the dimensionless representation of the temperature difference between the two horizontal boundaries of the layer. In many experimental situations, however, as well as in geophysical or astrophysical convection problems, the temperature difference is not a directly given parameter. In most of these cases the heat flux, or more exactly, the rate of heat production in a layer below the convection layer, has to be regarded as the given independent parameter of the problem. This does not cause any difference as long as the relation between heat flux and Rayleigh number is purely monotonic. In the presence of rotation, however, or when the temperature dependence of the material properties is considered, Rayleigh number and heat flux are not always monotonically related, and the physical situation differs with the heat production as the prescribed parameter instead of the temperature difference. In the following we will investigate this problem by proceeding from general discussions to a special case in which the calculations can be carried out explicitly.

First we shall develop a method similar to that used by Stuart (1958) to describe the time dependence of finite amplitude convection. This method considers the deviations from the static state of the layer introduced by the convection as small quantities. It is limited, therefore, to the neighbourhood of the critical Rayleigh number at which the static state becomes unstable.

Next we will show under very general assumptions that a stationary solution is unstable in the neighbourhood of the critical Rayleigh number when the heat transport increases with decreasing Rayleigh number. This can lead to the situation where a stable stationary solution does not exist for all values of the heat flux, even when there are stable stationary solutions for all Rayleigh numbers. This point raises the question as to the type of non-stationary convection by which the heat is transported in those cases. We will show that a periodic relaxation process occurs under certain conditions. The convective motion grows until the effective Rayleigh number is lowered so far that the convection dies away. As soon as the heat transport by conduction only has increased the temperature gradient, convection will start growing again.

Processes of the same physical nature occur in many problems. For example, the transport of stress by turbulent motion in parallel shear flow shows a similar behaviour known as the intermittency effect. Thus, the periodic convection may be useful as a model for transport phenomena in which the physical mechanism has unstable properties.

To discuss the problem in detail we must solve the equations of convection in the presence of a given rate of heat production in a conduction layer below the convection layer. We will start with the stationary solution of the static problem without convection. We then introduce the Rayleigh number corresponding to the temperature difference in the static solution as the parameter and treat the effects due to the time dependence of the temperature at the boundaries separately. In general, the problem of given heat production leads to a complex time-dependent boundary-value problem. The essential features of this problem, however, can be exhibited by considering two limit cases which show that convection with periodic time dependence can occur besides the asymptotically stationary solution.

2. Time dependence of convection

To describe the method of solution for time-dependent finite amplitude convection we will start with an equation of the following general form:

$$D_{\lambda\kappa} v_\kappa + R W_{\lambda\kappa} v_\kappa = Q_{\lambda\mu\nu} v_\mu v_\nu + U_{\lambda\mu}(\partial v_\kappa / \partial t), \tag{2.1}$$

where $D_{\lambda\kappa}$, $Q_{\lambda\mu\nu}$, $W_{\lambda\kappa}$ and $U_{\lambda\kappa}$ are time-independent differential matrix operators with respect to the spatial co-ordinates. R is the Rayleigh number, the relevant parameter of the problem, which corresponds to a characteristic temperature gradient in the fluid. Other dimensionless numbers, which may correspond to an imposed magnetic field or the rate of rotation in a rotating frame of reference, have not been noted explicitly. The vector v_κ denotes the deviation of the physical quantities such as temperature, components of the fluid velocity or the magnetic field from the basic state, on which the convection acts as a perturbation. We have chosen the general notation in (2.1) because our discussion is based only on some general properties of this equation and does not depend on the kind of variables included in v_κ. A wide class of convection problems can be written in the form (2.1) by using the index notation [for example, see Schlüter, Lortz & Busse (1965) and Lortz (1965)]. To complete the problem, proper boundary conditions for v_λ and its spatial derivatives have to be assumed. Since the vanishing solution always has to be a possible solution of the problem, the boundary conditions have to become linear homogeneous in the case of infinitesimal amplitudes. In this case $Q_{\lambda\mu\nu}$ can be omitted in (2.1) and we obtain a linear homogeneous boundary-value problem, which for simplicity we will call the linear problem. To exhibit the typical features of convection problems, we make the following assumptions about the linear problem.

(i) The linear problem has only exponentially decaying solutions, unless the Rayleigh number exceeds a critical value R_c. There exists at least one exponentially growing solution for $R > R_c$, whose growth rate depends continuously on the Rayleigh number R so that there exists a stationary solution at $R = R_c$.

(ii) There exists the adjoint problem for the stationary linear problem which is given by

$$D^+_{\lambda\kappa} u^+_\kappa + R W^+_{\lambda\kappa} u^+_\kappa = 0, \tag{2.2}$$

and proper boundary conditions so that the relation

$$\langle u''_\lambda, D_{\lambda\kappa} u'_\kappa\rangle + R\langle u''_\lambda, W_{\lambda\kappa} u'_\kappa\rangle = \langle u'_\lambda, D^+_{\lambda\kappa} u''_\kappa\rangle + R\langle u'_\lambda, W^+_{\lambda\kappa} u''_\kappa\rangle \tag{2.3}$$

holds, where u'_λ and u''_λ are arbitrary functions satisfying the boundary conditions of the linear problem and its adjoint problem respectively. The brackets indicate the average over the contained fluid.

(iii) In addition we will assume that $W_{\lambda\kappa}$ is not singular, so that we can normalize the eigenfunctions $u_{\lambda n}$ of the linear stationary problem regarding it as an eigenvalue problem with R as eigenvalue

$$\langle u^+_{\lambda n}, W_{\lambda\kappa} u_{\kappa m}\rangle = \delta_{nm}. \tag{2.4}$$

We are interested in solutions with small but finite amplitudes corresponding to Rayleigh numbers in the neighbourhood of the critical Rayleigh number R_c. Thus we try to obtain solutions by introducing the expansions

$$\left.\begin{aligned} v_\lambda &= \epsilon v^{(1)}_\lambda + \epsilon^2 v^{(2)}_\lambda + \ldots, \\ R &= R_c + \epsilon R^{(1)} + \epsilon^2 R^{(2)} + \ldots, \end{aligned}\right\} \tag{2.5}$$

where the time-independent parameter ϵ is a measure of the order of magnitude of the amplitude. This kind of expansion has been used by Malkus & Veronis (1958) and by Schlüter *et al.* (1965) to obtain solutions of the stationary problem. We wish to include time-dependent solutions and therefore have to relate the order of magnitude of the term $U_{\lambda\kappa}\,\partial v_\kappa/\partial t$ to the rest of the terms. For this purpose we scale the time, similarly to Veronis (1959), by introducing a 'relaxation frequency' μ,

$$t' = \mu t,$$

so that the maximum of $U_{\lambda\kappa}\,\partial v_\kappa/\partial t'$ becomes of the order of the term $W_{\lambda\kappa} v_\kappa$. The frequency μ then is expanded in the same way as R:

$$\mu = \mu^{(0)} + \epsilon\mu^{(1)} + \epsilon^2\mu^{(2)}. \tag{2.6}$$

Since we are interested in finite amplitude convection, we look for solutions whose amplitudes are bounded for all times, but not asymptotically decaying. This restriction will be sufficient to determine the time scale μ of the finite amplitude convection.

Introducing the expansion (2.5), (2.6) into (2.1), we obtain from the first order of ϵ the linear problem

$$D_{\lambda\kappa} v^{(1)}_\kappa + R_c W_{\lambda\kappa} v^{(1)}_\kappa = \mu^{(0)} U_{\lambda\kappa}\,\partial v^{(1)}_\kappa/\partial t'. \tag{2.7}$$

Since we have already assumed that there can exist only exponentially decaying solutions besides the stationary problem, we have to choose $\mu^{(0)} = 0$. The solution of this expansion can be written in the form

$$v_\kappa^{(1)} = u_\kappa(\mathbf{x})\,\alpha(t),$$

where the time dependence of $\alpha(t)$ is of higher order. Because the amplitude is already measured by ϵ, we will normalize this solution by subjecting it to the conditions

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \alpha(t)\,dt = 1 \tag{2.8}$$

and

$$\langle u_\lambda^+, W_{\lambda\kappa}\, u_\kappa\rangle = 1, \tag{2.9}$$

u_λ^+ is the solution of the adjoint problem corresponding to u_λ. Using the orthonormal set of eigenfunctions which belongs to the eigenvalue R_c we can describe this correspondence by

$$u_\lambda = \sum_n c_n\, u_{\lambda n}, \quad u_\lambda^+ = \sum_n c_n\, u_{\lambda n}^+.$$

From the orders ϵ^{n+1}, $n \geqslant 1$, we obtain linear inhomogeneous equations:

$$\begin{aligned} D_{\lambda\kappa}\, v_\kappa^{(n+1)} + R_c\, W_{\lambda\kappa}\, v_\kappa^{(n+1)} = {} & Q_{\lambda\mu\nu}(v_\mu^{(1)} v_\nu^{(n)} + \ldots + v_\mu^{(n)} v_\nu^{(1)}) \\ & - W_{\lambda\kappa}(R^{(1)} v_\kappa^{(n)} + \ldots + R^{(n)} v_\kappa^{(1)}) \\ & + U_{\lambda\kappa}\,\partial(\mu^{(1)} v_\kappa^{(n)} + \ldots + \mu^{(n)} c_\kappa^{(1)})/\partial t'. \end{aligned} \tag{2.10}$$

Since the inhomogeneous part of each equation is known from the solutions of the lower-order equations, this system (2.10) of equations can be solved sequentially. The only unknown terms $-W_{\lambda\kappa}\, R^{(n)} v_\kappa^{(1)} + U_{\lambda\kappa}\,\mu^{(n)}\,\partial v_\kappa^{(1)}/\partial t'$ are determined by the solvability condition. In order for a solution to exist, the inhomogeneous part on the right side of (2.10) has to be orthogonal to all solutions of the adjoint linear homogeneous equation. These solutions are represented by the set $u_{\lambda n}^+$ and thus the condition of solvability yields, for example, in second order the following system of equations:

$$\langle u_{\lambda n}^+, Q_{\lambda\mu\nu}\, v_\mu^{(1)} v_\nu^{(1)}\rangle - R^{(1)}\langle u_{\lambda n}^+, W_{\lambda\kappa}\, v_\kappa^{(1)}\rangle + \langle u_{\lambda n}^+, U_{\lambda\kappa}\mu^{(1)}\,\partial v_\kappa^{(1)}/\partial t'\rangle = 0. \tag{2.11}$$

If we choose $\alpha(t')\, u_\lambda^+$ instead of $u_{\lambda n}^+$, we get a necessary condition which will be sufficient to determine the time dependence in this order. To discuss this condition we will assume at first that the first term in (2.11) vanishes identically. In this case we obtain

$$M\mu^{(1)}\,\partial\alpha^2(t')/\partial t' - R^{(1)}\alpha^2(t') = 0, \tag{2.12}$$

where M is the constant $\frac{1}{2}\langle u_\lambda^+, U_{\lambda\kappa}\, u_\kappa\rangle$. Because this equation again admits only exponentially time-dependent solutions for $\alpha(t')$, we have to conclude

$$\mu^{(1)} = R^{(1)} = 0$$

in this case. The higher orders will yield the same result as long as the term with $Q_{\lambda\mu\nu}$ vanishes identically. To discuss the general case, we will assume that in the $(n+1)$th order we first get a non-vanishing integral term with $Q_{\lambda\mu\nu}$. The equation for the amplitude then takes the form

$$\langle u_\lambda, Q_{\lambda\mu\nu}(v_\mu^{(1)} v_\nu^{(n)} + \ldots + v_\mu^{(n)} v_\nu^{(1)})\rangle\,\alpha(t') - R^{(n)}\,\alpha^2(t') + \mu^{(n)} M\,\partial\alpha^2(t')/\partial t' = 0. \tag{2.13}$$

Since we can replace the bracket by $Q\alpha^{n+1}(t')$ with a time-independent constant Q, the solution of this equation corresponding to an initial value α_0 at the time $t' = 0$ can be obtained easily:

$$\alpha(t') = \frac{\alpha_0 \exp\left[nR^{(n)}t'/M\mu^{(n)}\right]}{1+\alpha_0 Q(\exp\left[nR^{(n)}t'/M\mu^{(n)}\right]-1)/R^{(n)}}. \tag{2.14}$$

This solution includes the case of the stationary finite amplitude

$$\alpha(t') = \alpha_0 = R^{(n)}/Q,$$

where $R^{(n)}$ is determined by $R^{(n)} = Q$ according to the normalization condition (2.9). The other solutions approach the stationary solution for $t > 0$ if $\epsilon^n R^{(n)}$ is positive. If $\epsilon^n R^{(n)}$ is negative every solution corresponding to an initial value α_0 different from 1 is either decaying to zero or diverging. Thus the stationary solution is unstable in this case with respect to perturbations of the amplitude. Since for sufficiently small amplitudes $|\epsilon|$ the Rayleigh number is given by $R = R_c + \epsilon^n R^{(n)}$, we can formulate the result: *every stationary solution is unstable in a range of sufficiently small amplitudes* $|\epsilon|$ *for which* $(d/d|\epsilon|)(R-R_c)$ *is negative.* There are convection problems where due to this kind of instability all stationary solutions are unstable for sufficiently small amplitudes $|\epsilon|$. Since the heat transport is usually a monotone function of the amplitude $|\epsilon|$, this leads to the question: which kind of convection will occur, if a fixed value of the heat flux is given corresponding to amplitudes in this range?

Before we discuss the problem of convection in the case of a given heat flux, we have to make several remarks on the subject of this section.

First we will rewrite (2.13) after multiplying it by ϵ^{n+2}. Introducing the definition $\epsilon^2\alpha^2(t) = A(t)$ and using the approximate relations

$$\mu^{(n)}\epsilon^{(n)}t \approx t', \quad \epsilon^n R^{(n)} \approx R - R_c,$$

we obtain

$$M\,dA(t)/dt = A(t)\left[R - R_c - R^{(n)}A(t)^{\frac{1}{2}n}\right]. \tag{2.15}$$

In some cases the value of $R^{(n)}$ is very small, so that the contributions of the next higher order may be of equal importance even for rather small amplitudes $|\epsilon|$, where the contributions of the remaining higher orders still can be neglected. Taking, for example, $n = 1$ we can include the effect of the next higher order by writing instead of (2.15)

$$M\,dA(t)/dt = A(t)\left[R - R_c - R^{(1)}A^{\frac{1}{2}}(t) - R^{(2)}A(t)\right], \tag{2.16}$$

corresponding to the approximation $R - R_c \approx \epsilon R^{(1)} + \epsilon^2 R^{(2)}$ for the Rayleigh number. Equation (2.16) describes the time dependence of the solution, which represents the convection flow in the form of hexagonal cells in a fluid layer heated from below. In this case $R^{(1)}$ can be regarded as small because it is proportional to the variation of the material properties in the layer. The hexagon solution has been investigated by several authors (for a review of this work see Segel, 1966). An analysis which includes the variation of all material properties has been done by Busse (1962), to which we will hereafter refer as I. It has been shown in I that the hexagon solution can be stable only if $\epsilon R^{(1)}$ is negative. Hence this solution is an example of the instability discussed in this section. Figure 1 *a* shows qualitatively the function $\epsilon(R)$ indicating the unstable range by a dashed line. The corresponding dependence of the heat transport is plotted in figure 1*b* assuming $R^{(2)} < 1$. Since it has also been shown in I that no other stable solution

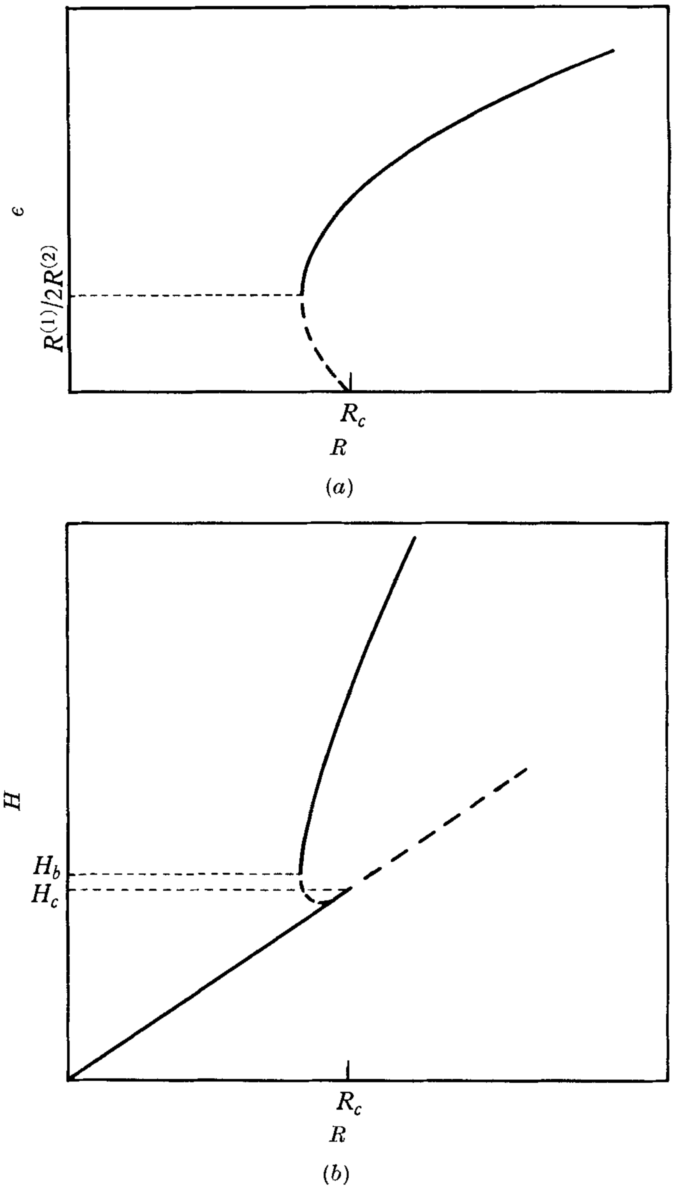

FIGURE 1. (*a*) The dependence of the amplitude ϵ on the Rayleigh number R. (*b*) The dependence of the heat transport H on the Rayleigh number R.

exists in a certain neighbourhood of R_c, the heat flux H shows the range $H_c < H < H_b$ where it cannot be transported by a stationary solution. This fact has not yet been proved for other problems with unstable $H(R)$ dependence. Therefore, we will use in further discussion the case of the hexagon solution to exhibit the features of non-stationary convection. When $R^{(2)}$ becomes greater than 1, as in the case where the Prandtl number is small in comparison with 1, the point H_b drops below H_c, and stationary solutions are possible for all values of H in the range where the theory is applicable.

3. Boundary conditions for a convection layer with given rate of heat production

In this section we will assume that the convection layer is confined between two horizontal infinite planes. We introduce Cartesian co-ordinates with the z-direction perpendicular to the planes so that the convection layer is described by

$$-\tfrac{1}{2} \leqslant z \leqslant \tfrac{1}{2}.$$

Adjacent to the convection layer we assume two solid conductive layers, the lower one confined to the region

$$-\tfrac{1}{2} - D \leqslant z \leqslant -\tfrac{1}{2}.$$

For simplicity we will assume that the upper conduction layer is completely symmetric with the lower one with respect to the plane $z = 0$. Further, we assume that the conduction layers have constant heat conductivity and constant heat capacity per unit volume differing from the values of the convection layer by the factors ξ and η. In order that the static state may be a possible solution of the problem, the heat sources in the lower conduction layer and the corresponding heat sinks in the upper layer have to be arranged homogeneously with respect to the horizontal co-ordinates. Thus the temperature distribution T_0 in the static case depends only on z, and we can write the temperature distribution

$$T(x,y,z,t) = T_0(z) + \theta(x,y,z,t). \tag{3.1}$$

Our intention is to find boundary conditions for the deviation θ from the static solution in such a way that the problem is reduced to a boundary-value problem for the convection layer alone. Since the heat production is constant the equation for θ in the upper conduction layer is linear and homogeneous,

$$\xi\Delta\theta - \eta\, \partial\theta/\partial t = 0. \tag{3.2}$$

Only ξ and η occur in this equation since we assume that all variables have been made dimensionless in terms of characteristic quantities of the convection layer in the usual procedure (Schlüter *et al.* 1965). The solution of (3.2) must fulfil the conditions of continuity of temperature and heat flux at $z = \frac{1}{2}$ and a boundary condition at $z = D + \frac{1}{2}$. Assuming that the whole system is enclosed by isolating walls the latter condition is

$$\partial\theta/\partial z = 0 \quad \text{at} \quad z = D + \tfrac{1}{2}. \tag{3.3}$$

We will solve (3.2) in terms of a given boundary value θ_B at $z = \frac{1}{2}$. The condition of continuity for the heat flux,

$$(\partial\theta/\partial z)|_{z=\frac{1}{2}+0} = (\partial\theta/\partial z)|_{z=\frac{1}{2}-0} \equiv (\partial\theta/\partial z)_B, \tag{3.4}$$

will then give a relation between θ_B and $(\partial\theta_B/\partial t)$, which is the boundary condition for the temperature in the convection layer.

We can assume that θ_B satisfies the equation

$$\Delta_2 \theta_B = -a^2\theta_B, \tag{3.5}$$

where Δ_2 is the Laplacian with respect to the horizontal dimensions. This assumption allows us to separate variables in (3.2) and does not restrict this linear problem because an arbitrary θ_B can be represented by a sum of trigonometric functions which satisfy an equation of the form (3.5).

Using the function

$$\chi(z,t) = \frac{\cosh a(D+\frac{1}{2}-z)}{\cosh aD} - \frac{2\pi}{D^2}\exp\{-a^2\xi t/\eta\}$$
$$\times \sum_{n=0}^{\infty} \frac{(n+\frac{1}{2})\sin\{\pi(n+\frac{1}{2})(z-\frac{1}{2})/D\}}{a^2+(n+\frac{1}{2})\pi^2/D^2}\exp\{-(n+\tfrac{1}{2})^2\pi^2\xi t/D^2\eta\}$$

which is the solution of the following problem (see Courant & Hilbert 1937):

$$(\partial^2/\partial z^2 - a^2 - (\eta/\xi)\,\partial/\partial t)\,\chi = 0 \quad \text{with} \quad \left\{\begin{aligned} (\partial/\partial z)\,\chi(D+\tfrac{1}{2},t) &= 0,\\ \chi(\tfrac{1}{2},t) &= 1 \text{ for } t<0,\\ \chi(z,0) &= 0,\end{aligned}\right\} \tag{3.6}$$

we can write the solution of (3.2) in the form

$$\theta = \int_0^t \frac{\partial\theta_B}{\partial\tau}\,\chi(z,t-\tau)\,d\tau, \tag{3.7}$$

with $\theta_B(t) = 0$ for $t < 0$ as the initial condition.

With (3.7) the boundary condition for the temperature θ in the convection layer can be written immediately:

$$\frac{\partial\theta}{\partial z} = \pm\xi\int_0^t \frac{\partial\theta}{\partial\tau}\frac{\partial}{\partial z}[\chi(z,t-\tau)]_{z=\frac{1}{2}}\,d\tau \quad \text{at} \quad z = \pm\tfrac{1}{2}. \tag{3.8}$$

Since this general form of the boundary condition is very complicated, we will discuss it in two special cases where (3.8) can be simplified. These are case A when the thermal thickness $(\eta|\xi)^{\frac{1}{2}}D$ of the conduction layer is small as compared with the thickness of the convection layer, and case B when the thickness of the conduction layer is large.

Since we know from the discussion in § 2 that the time variation of the convection can be neglected to the order of the linear part of the equation, we can assume in case A where $(\eta|\xi)^{\frac{1}{2}}D \ll 1$ holds, that the time dependence of the term $(\partial\theta_B/\partial\tau)$ is small as compared with the time dependence of the exponential terms in $\chi(z,t-\tau)$. Thus the evaluation of the integral (3.8) yields

$$\frac{\partial\theta}{\partial z} = \pm\xi\left[-\theta a\tanh aD - \frac{\partial\theta}{\partial t}\frac{2\eta}{D\xi}\sum_{n=0}^{\infty}\frac{(n+\frac{1}{2})^2(\pi/D)^2}{(a^2+(n+\frac{1}{2})^2(\pi/D)^2)^2}\right]. \tag{3.9}$$

It is known from the solutions of the stationary convection problem that the parameter a is either vanishing or is of the order π. In the latter case the second term in (3.9) has to be neglected in comparison with the first term because of the slight time dependence of θ. For $a = 0$ the first term vanishes and we obtain as boundary conditions in case A:

$$\partial\theta/\partial z = \mp a\xi\theta \tanh aD \quad \text{at} \quad z = \pm\tfrac{1}{2} \quad \text{for} \quad a \neq 0, \tag{3.10a}$$

$$\partial\theta/\partial z = \mp \eta D\, \partial\theta/\partial t \quad \text{at} \quad z = \pm\tfrac{1}{2} \quad \text{for} \quad a = 0. \tag{3.10b}$$

In case B, when D is large as compared to 1, it is convenient to assume the limit case $D = \infty$, where (3.7) takes the integral form

$$\theta = \int_0^t \frac{\partial\theta_B}{\partial\tau}\Big[\exp\{-a(z-\tfrac{1}{2})\} - 2/\pi \exp\{-a^2\xi(t-\tau)/\eta\} \times \int_0^\infty \frac{\zeta \sin(z-\frac{1}{2})\zeta}{a^2+\zeta^2}\exp\{-\zeta^2\xi(t-\tau)/\eta\}\,d\zeta\Big]\,d\tau. \tag{3.11}$$

By transforming the integral we arrive at the following boundary condition in case B:

$$\frac{\partial\theta}{\partial z} = \mp\left(\frac{\xi\eta}{\pi}\right)^{\frac{1}{2}}\exp\{-\xi a^2 t/\eta\}\int_0^t \frac{\partial(\theta\exp\{\xi a^2\tau/\eta\})/\partial\tau}{(t-\tau)^{\frac{1}{2}}}\,d\tau. \tag{3.12}$$

Again we can make the distinction between those components of temperature whose average values with respect to the horizontal dimensions vanish and for which the exponential time dependence is dominant in (3.12) in comparison with that of θ, and the component that gives the average value of θ and corresponds to $a = 0$. Hence the boundary conditions in case B are:

$$\left.\begin{aligned} \frac{\partial\theta}{\partial z} &= \mp a\xi\theta && \text{at } z = \pm\tfrac{1}{2} \quad \text{for} \quad a \neq 0,\\ \frac{\partial\theta}{\partial z} &= \mp(\xi\eta)^{\frac{1}{2}}\frac{1}{\sqrt{\pi}}\int_0^t \frac{(\partial\theta/\partial\tau)}{(t-\tau)^{\frac{1}{2}}}\,d\tau && \text{at } z = \pm\tfrac{1}{2} \quad \text{for} \quad a = 0. \end{aligned}\right\} \tag{3.13}$$

4. The equation for the time dependence of the amplitude

The boundary conditions for the temperature θ in the convection layer show the fact that, in case A as well as in case B, only the condition for the horizontal average of θ, which we will denote by $\overline{\theta}$, is time dependent. We will assume in this section that the solution of the stationary problem with fixed values of the average temperature at the boundary, corresponding to the boundary condition $\overline{\theta} = 0$ at $z = \pm\frac{1}{2}$, is known. Actually this is true only for the case of infinite conducting boundaries, where ξ goes to infinity in (3.10) and (3.13) and explicit calculations are given in I for various boundary conditions of the fluid velocity. It has been shown, however, in I that the case of finitely conducting boundaries leads qualitatively to the same results. Since our considerations do not depend upon the exact values of $R^{(1)}$ and $R^{(2)}$, the assumption is appropriate.

Because $\overline{\theta}$ vanishes in the lowest order—the static state is always stable with respect to disturbances which do not depend on the horizontal dimensions—we have to deal with the equation only in the second order where in the heat conduction equation a term due to the convection arises:

$$\Delta\overline{\theta}^{(2)} - \partial\overline{\theta}^{(2)}/\partial t = \overline{v_j^{(1)}\,\partial_j\,\theta^{(1)}}. \tag{4.1}$$

In this equation v_j is the vector of the convection flow. The time derivative can be neglected, corresponding to the result $\mu^{(0)} = 0$ in § 2. In order to use the known results of the stationary solution, we make the definition

$$\bar{\theta}^{(2)}(z,t) = 2zh(t) + \hat{\bar{\theta}}^{(2)}(z,t), \tag{4.2}$$

where $\hat{\bar{\theta}}^{(2)}$ satisfies the equation

$$\Delta\hat{\bar{\theta}}^{(2)} - \partial\hat{\bar{\theta}}^{(2)}/\partial t = v_j^{(1)}\,\partial_j\,\theta^{(1)}, \tag{4.3}$$

with the boundary condition $\hat{\bar{\theta}}^{(2)} = 0$ at $z = \pm\frac{1}{2}$. Thus $\hat{\bar{\theta}}^{(2)}$ becomes identical to the corresponding solution of the stationary problem multiplied by the factor $\alpha^2(t)$.

In case A the boundary condition (3.10*b*) yields for $h(t)$ the equation

$$\eta D\,dh/dt = -2h - \partial\bar{\theta}^{(2)}/\partial z|_{z=\pm\frac{1}{2}}, \tag{4.4}$$

which is solved by

$$h(t) = \frac{-1}{\eta D}\int_0^t \exp\left\{\frac{-2(t-\tau)}{\eta D}\right\}\partial\hat{\bar{\theta}}^{(2)}/\partial z|_{z=\pm\frac{1}{2}}\,d\tau. \tag{4.5}$$

In case B the equation

$$\left(\frac{\xi\eta}{\pi}\right)^{\frac{1}{2}}\int_0^t \frac{dh/d\tau}{(t-\tau)^{\frac{1}{2}}}\,d\tau = -2h - \partial\hat{\bar{\theta}}^{(2)}/\partial z|_{z=\pm\frac{1}{2}} \tag{4.6}$$

has the implicit solution

$$h(t) = \frac{-1}{(\pi\eta\xi)^{\frac{1}{2}}}\int_0^t \frac{2h + \partial\hat{\bar{\theta}}^{(2)}/\partial z|_{z=\pm\frac{1}{2}}}{(t-\tau)^{\frac{1}{2}}}\,d\tau. \tag{4.7}$$

The term by which $\bar{\theta}^{(2)}$ enters the solvability condition in the third order is given by

$$\langle\theta^{(1)}, v_j^{(1)}\,\partial_j\,\bar{\theta}^{(2)}\rangle = \langle\theta^{(1)}, v_z^{(1)}\,\partial_z\,\bar{\theta}^{(2)}\rangle. \tag{4.8}$$

Since $\epsilon^2\langle\theta, v_z\rangle$ is the average of the convective heat transport which by definition is equal to $(\partial\bar{\theta}/\partial z)|_{z=\pm\frac{1}{2}}$ and because the normalization (2.9) leads to

$$\langle\theta^{(1)}v_z^{(1)}\rangle = \alpha^2(t) \tag{4.9}$$

in this special problem, we can write the additional term in the solvability condition

$$\left.\begin{aligned}\langle\theta^{(1)}, v_z^{(1)}\,\partial 2zh(t)/\partial z\rangle &= \alpha^2(t)\,2h(t)\\ &= \alpha^2(t)\int_0^t \frac{2}{\eta D}\exp\left\{-\frac{2}{\eta D}(t-\tau)\right\}\alpha^2(\tau)\,d\tau \quad \text{in case A,}\\ &= \alpha^2(t)\int_0^t \frac{\alpha^2(\tau) - 2h(\tau)}{(\pi\xi\eta)^{\frac{1}{2}}(t-\tau)^{\frac{1}{2}}}\,d\tau \qquad\qquad \text{in case B.}\end{aligned}\right\} \tag{4.10}$$

Now the equation for the time dependence of the amplitude in the case of given heat production can be derived from the corresponding equation (2.16) in the case of fixed average temperature at the boundary by adding simply the term (4.10),

$$dA(t)/dt = A(t)\,[R - R_c - R^{(2)}A(t) + R^{(1)}A^{\frac{1}{2}}(t) - 2h(t\,.\,M)]. \tag{4.11}$$

To eliminate the factor M we have used $t' = t/M$ as the time variable and dropped the prime after the transformation. In addition we have changed the sign in front of $R^{(1)}$ regarding henceforth both $R^{(1)}$ and $A^{\frac{1}{2}}(t)$ as positive quantities. In § 5 we will discuss this equation in the case A showing that the instability of the stationary solution leads to periodic time-dependent solutions. The discussion of case B in § 6 will yield the result that, in this case, the stationary solution is stabilized for all amplitudes.

5. Periodic time-dependent convection

Before discussing (4.11) in case A we transform it by differentiation into an ordinary differential equation using the expression (4.10) for $h(t)$,

$$\ddot{A} - \frac{\dot{A}\dot{A}}{A} + (\kappa + R^{(2)}A - \tfrac{1}{2}R^{(1)}A^{\frac{1}{2}})\,\dot{A} - \kappa A(R - R_c - R^{(2)}A + R^{(1)}A^{\frac{1}{2}} - A) = 0. \quad (5.1)$$

The dot indicates the differentiation with respect to the time and κ is an abbreviation for $2M/\eta D$. This equation has the stationary solution

$$A_s^{\frac{1}{2}} = \frac{R^{(1)}}{2(R^{(2)}+1)} \pm \left[\frac{R^{(1)2}}{4(R^{(2)}+1)^2} + \frac{R-R_c}{R^{(2)}+1}\right]^{\frac{1}{2}}. \quad (5.2)$$

To determine when this solution is stable we superpose an infinitesimal disturbance with time dependence $\exp(\sigma t)$ and obtain the following relation for σ from (5.1):

$$\begin{aligned}\sigma = {} & \tfrac{1}{2}(\tfrac{1}{2}R^{(1)}A_s^{\frac{1}{2}} - R^{(2)}A_s - \kappa) \\ & \pm (\tfrac{1}{4}(\tfrac{1}{2}R^{(1)}A_s^{\frac{1}{2}} - R^{(2)}A_s - \kappa)^2 - \kappa(A_s[R^{(2)}+1] - \tfrac{1}{2}R^{(1)}A_s^{\frac{1}{2}}))^{\frac{1}{2}}. \end{aligned} \quad (5.3)$$

In the range of amplitude $A_s^{\frac{1}{2}} \leqslant R^{(1)}/2(R^{(2)}+1)$, i.e. in the range where in (5.2) the negative sign is valid, the radical in (5.3) is always greater in magnitude than the first term and growth rates of both signs exist. The range of this instability as well as the dependence of the amplitude $A_s^{\frac{1}{2}}$ on the Rayleigh number R is the same as for the solution with given temperature difference, which is plotted in figure 1(*a*). Only the scale is different since $R^{(2)}$ in that figure has to be replaced now by $R^{(2)}+1$.

In the range $A_s^{\frac{1}{2}} \geqslant R^{(1)}/(2R^{(2)}+2)$ instability occurs if, and only if, the first term on the right side of (5.3) is positive. This condition for instability

$$\tfrac{1}{2}R^{(1)}A_s^{\frac{1}{2}} - R^{(2)}A_s - \kappa > 0 \quad (5.4)$$

shows that κ has a stabilizing effect. Only in the limit $\kappa \to 0$ the stationary solution is unstable in the entire range $A_s^{\frac{1}{2}} < R^{(1)}/2R^{(2)}$, in which the solution with fixed average temperature at the boundaries is unstable. Using (5.2) we can rewrite (5.4) in terms of given parameters of the problem

$$\tfrac{1}{4}(1-R^{(2)}) - \frac{R-R_c}{R^{(1)2}}R^{(2)2} > \kappa\frac{R^{(2)}+1}{R^{(1)2}}\,\frac{R^{(1)} + R^{(2)}(R^{(1)2}+4(R^{(2)}+1)(R-R_c))^{\frac{1}{2}}}{R^{(1)} + (R^{(1)2}+4(R^{(2)}+1)(R-R_c))^{\frac{1}{2}}}. \quad (5.5)$$

In figure 2 the range of instability given by the inequality (5.5) is shown for different values of $\kappa/R^{(1)2}$. In the cases of infinite conducting boundaries with Prandtl numbers at least of the order one, $R^{(2)}$ for the hexagon solution is less than one whatever the boundary conditions for the fluid velocity vector are. (For values of $R^{(1)}$ and $R^{(2)}$ see I; values for $R^{(2)}$ are given also by Malkus & Veronis (1958) and Schlüter *et al.* (1965).) For boundaries with finite conductivity and therefore with less restraint, smaller values of $R^{(2)}$ should be expected. When the Prandtl number is small in comparison with 1, $R^{(2)}$ exceeds 1 and instability can appear for subcritical values of R only. In all cases the stationary solution can be stabilized if κ becomes sufficiently large according to solution (5.4).

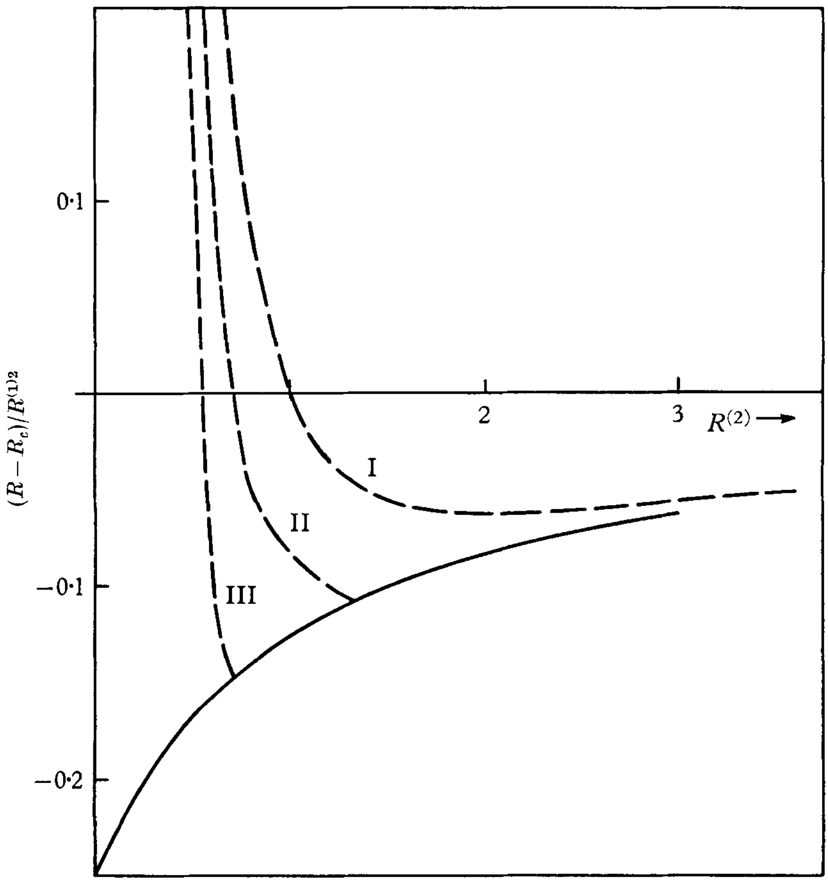

FIGURE 2. The instability region. The stationary solution $A_s^{\frac{1}{2}}$ exists for parameter values above the solid line. It is unstable below the dashed lines corresponding to the values 0 (I), 0·05 (II), 0·1 (III) of $\kappa/R^{(1)2}$.

In discussing the stability range we have assumed that all other disturbances besides the perturbation of the amplitude are irrevelant. This is justified because the stability analysis in I does not change, since no term with $\bar{\theta}$ appears in the expressions for the growth rates of those disturbances.

In order to discuss the general time-dependent solution of (5.1) we transform (5.1) by introducing $X = \ln A$,

$$\ddot{X} + (\kappa + R^{(2)}e^{X} - \tfrac{1}{2}R^{(1)}e^{\frac{1}{2}X})\,\dot{X} + \kappa(e^{X} + R^{(2)}e^{X} - R^{(1)}e^{\frac{1}{2}X} - R + R_c) = 0. \quad (5.6)$$

This equation has the form of a generalized Lienard differential equation. Several theorems about the existence of periodic solutions of the equation can be found in the literature. The theorem by Levinson & Smith (1942) can be applied in the case of (5.6). It states that a periodic solution of (5.6) exists when:

$$\left.\begin{array}{ll}\text{(i)} & \text{the last bracket in (5.6) has a zero,}\\ \text{(ii)} & \text{the first bracket is negative at this zero,}\\ \text{(iii)} & R-R_c > 0.\end{array}\right\} \tag{5.7}$$

Since the zero determines the stationary solution, condition (ii) is identical with condition (5.4) for the instability of the stationary solution. In their paper Levinson & Smith also give a sufficient condition for the stability of the periodic solution. In terms of the amplitude $A^{\frac{1}{2}}$ this condition is

$$R^{(2)}\bar{A}-\tfrac{1}{2}R^{(1)}\overline{A^{\frac{1}{2}}}+\kappa > 0, \tag{5.8}$$

where the bar indicates the average with respect to time. This condition includes condition (5.4) as a special case when the periodic solution is stationary. Hence we obtain a good description of the behaviour of the solution of (5.1): as long as the condition (5.8) can be fulfilled by the stationary solution, this solution is the stable solution; if this is not possible, the stationary solution changes into a periodic solution, where the condition (5.8) still can be fulfilled because the relation $\bar{A} > (\overline{A^{\frac{1}{2}}})^2$ holds.

For the numerical calculation we have used form (4.11) of the equation. Since the equation retains its form with $(R-R_c)/c$ in place of $R-R_c$ when the transformation

$$A' = A/c, \quad t' = ct, \quad \kappa' = \kappa/c, \quad R^{(1)'} = R^{(1)}/c^{\frac{1}{2}}$$

with constant c is applied, the dependence on one of the parameters can be neglected. The numerical integrations (examples are plotted in figures 3–6) confirm the description of the solution given above. According to (4.11) they show the following behaviour in the case $R > R_c$: starting with a small initial value the solution grows exponentially because the first term on the right side determines the behaviour. Later the rate of growth increases due to the third term until the second term becomes important. If κ is small, the solution comes close to the upper bound given by the vanishing right side of (4.11) for $\kappa = 0$:

$$A_m^{\frac{1}{2}} = \frac{R^{(1)}}{2R^{(2)}}+\left[\left(\frac{R^{(1)}}{2R^{(2)}}\right)+\frac{R-R_c}{R^{(2)}}\right]^{\frac{1}{2}}. \tag{5.9}$$

Meanwhile the integral term in (4.11) increases and the solution decreases with a speed approximately proportional to κ. When a stable stationary solution exists, this behaviour iterates in damped oscillations approaching the stationary solution asymptotically. If, however, the stationary solution is unstable, the decrease of the amplitude continues to very small, almost vanishing amplitudes. Only when the integral term has become smaller than $R-R_c$ can the solution start growing again. Because of this behaviour the period of the solution is approximately proportional to κ^{-1}.

For $R-R_c < 0$ the stationary solution is approached in damped oscillations in the case where it is stable and the initial value of $A^{\frac{1}{2}}$ fulfils the condition

$$A^{\frac{1}{2}} > \frac{R^{(1)}}{2R^{(2)}} - \left[\left(\frac{R^{(1)}}{2R^{(2)}}\right)^2 + \frac{R-R_c}{R^{(2)}}\right]^{\frac{1}{2}}. \tag{5.10}$$

Otherwise the stable solution of vanishing amplitude is reached asymptotically.

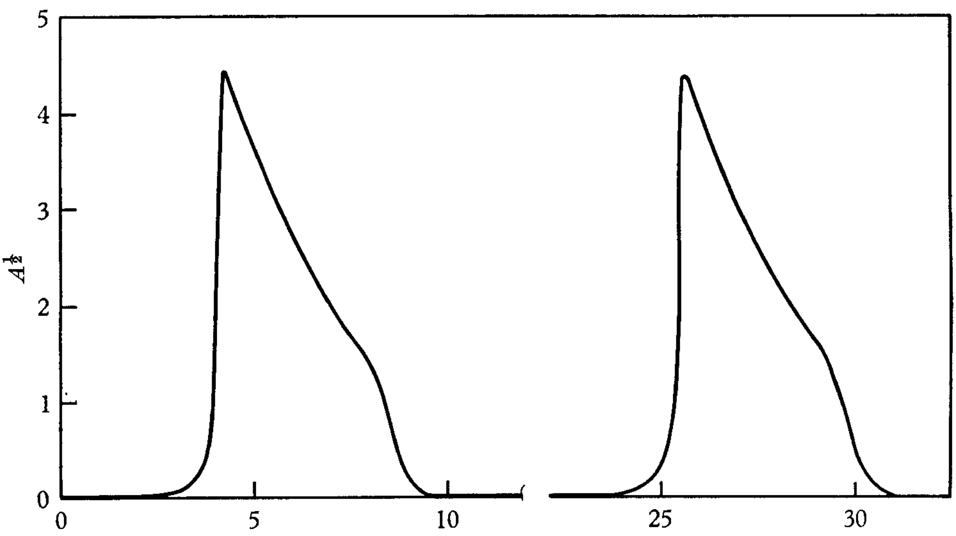

FIGURE 3. The time dependence of $A^{\frac{1}{2}}$ for $R-R_c = 1$; $R^{(1)} = 3$; $R^{(2)} = 0{\cdot}7$; $\kappa = 0{\cdot}05$.

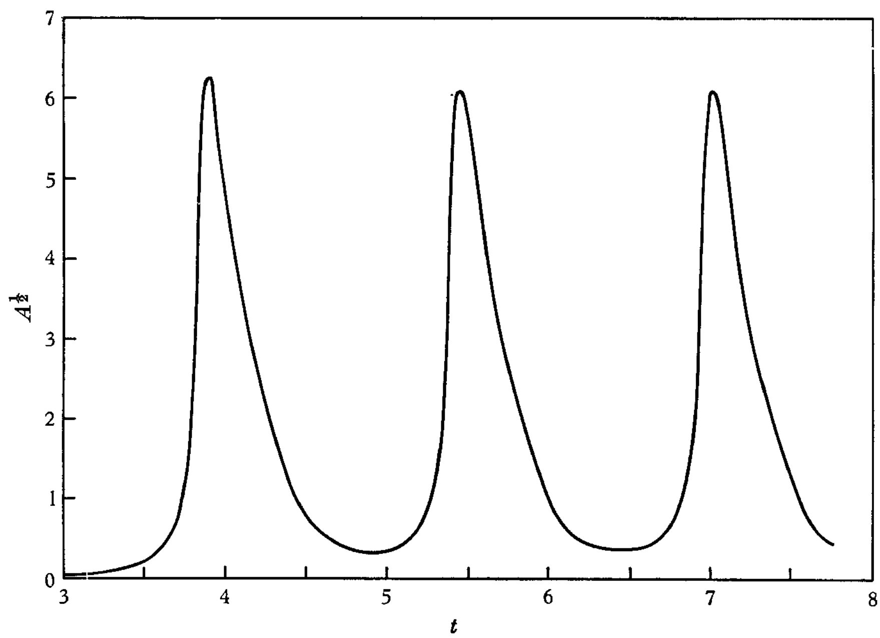

FIGURE 4. The time dependence of $A^{\frac{1}{2}}$ for $R-R_c = 1$; $R^{(1)} = 5$; $R^{(2)} = 0{\cdot}7$; $\kappa = 0{\cdot}5$.

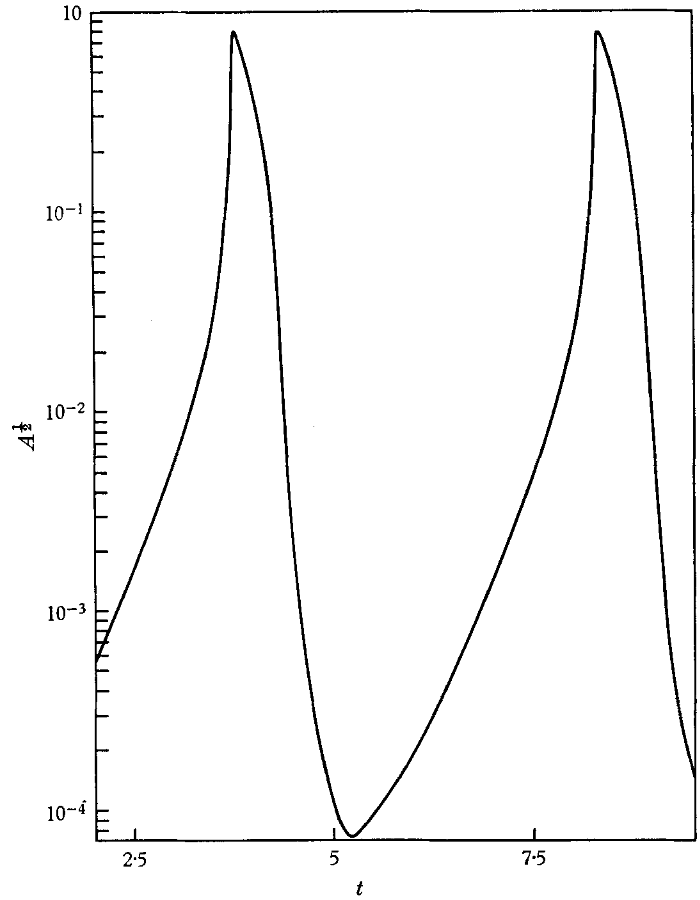

FIGURE 5. The time dependence of $A^{\frac{1}{2}}$ for $R-R_c = 1$; $R^{(1)} = 6$; $R^{(2)} = 0{\cdot}7$; $\kappa = 0{\cdot}5$.

6. Convection between two infinitely extended conduction layers

As in case A the stationary solution of (4.10) for $\epsilon^2 h(t)$ in case B is given by

$$\epsilon^2 h(t) = A_s. \tag{6.1}$$

Thus the expression (5.2) for the amplitude A_s of the stationary convection holds also in case B. Again we will discuss the stability of the stationary solution by superposing an infinitesimal disturbance $D(t)$. Since the equation for $D(t)$,

$$\dot{D}(t) = D(t)\,(\tfrac{1}{2}R^{(1)}A_s^{\frac{1}{2}} - R^{(2)}A_s) + A_s K(t), \tag{6.2}$$

with
$$K(t) = \frac{1}{\pi^{\frac{1}{2}}}\int_0^t \frac{D(\tau)-K(\tau)}{(t-\tau)^{\frac{1}{2}}}\,d\tau\,.\left(\frac{M}{\eta\xi}\right)^{\frac{1}{2}}, \tag{6.3}$$

does not allow solutions of exponential time dependence, we use the method of Laplace transformation. Indicating the transformed variables by a tilde,

$$\tilde{D} = \int_0^\infty e^{-st} D(t)\,dt, \tag{6.4}$$

we get the following equations for the transformed variables:

$$s\tilde{D}-\tilde{D}_0 = \tilde{D}(\tfrac{1}{2}R^{(1)}A_s^{\frac{1}{2}}-R^{(2)}A_s)+A_s\tilde{K}, \tag{6.5}$$

$$\tilde{K} = (-\tilde{K}+\tilde{D})\,(M/s\eta\xi)^{\frac{1}{2}}. \tag{6.6}$$

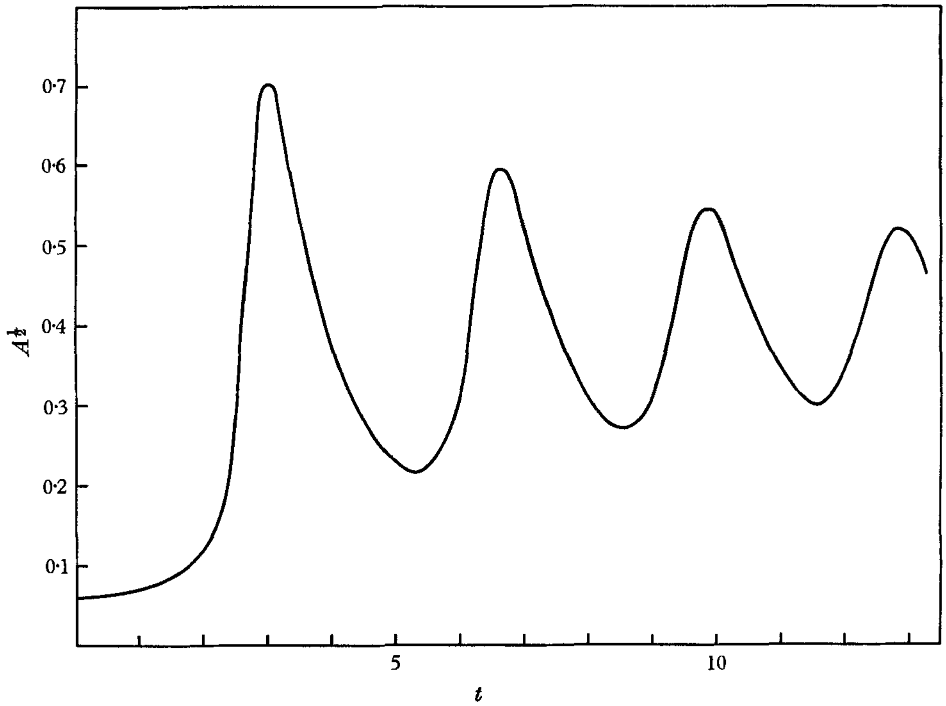

FIGURE 6. The time dependence of $A^{\frac{1}{2}}$ for $R-R_c = -0{\cdot}05$; $R^{(1)} = 1$; $R^{(2)} = 1{\cdot}1$; $\kappa = 0{\cdot}02$.

The solution of these equations is given by

$$\tilde{D} = \tilde{D}_0\left(s-\tfrac{1}{2}R^{(1)}A_s^{\frac{1}{2}}+R^{(2)}A_s+\frac{A_s}{(s\eta\xi/M)^{\frac{1}{2}}+1}\right)^{-1}. \tag{6.7}$$

Since the expression (6.7) has no pole in the right half of the complex plane if $A_s^{\frac{1}{2}} \geqslant R^{(1)}/2(R^{(2)}+1)$, the stationary solution is stable in this region. For smaller amplitudes the disturbance $D(t)$ has a growing time dependence. This instability corresponds to the positive growth rate (5.3) in the same region. It does not give rise to periodic time dependence but to solutions approaching either the vanishing or the stable stationary finite amplitude solution.

7. Conclusion

The concept of non-stationary convection frequently has been used to explain non-stationary processes in the earth's interior or in astrophysical problems. Periodic convection for Rayleigh numbers close to the critical value depends, according to § 5, mainly on the following conditions. First, the material properties of the layer have to be sufficiently non-symmetric with respect to the middle plane of the layer. $R^{(1)}$ will be of the order of $R^{(2)}$ if the variation of some material properties throughout the layer is of the order of their mean value. This can have the effect that only a part of the layer is gravitationally unstable in the static case. The second condition is a high enough heat capacity in the adjacent layer, which serves as a heat reservoir. In applications of this model, phase transitions can provide this capacity. A further condition for periodic convection is that the Prandtl number has to be of the order one or larger; otherwise the convective heat transport becomes unimportant in comparison with the conductive heat transport.

The physical ideas, however, on which we based the discussion of non-stationary convection are important in a much wider range of problems. Howard (1964) has shown that a similar process where a conduction layer serves as an energy reservoir can explain features of turbulent convection at high Rayleigh numbers. Since most of the problems where this physical process is present are very complex, it was the intention of this work to exhibit the characteristic features of this process in a special case which may serve as a model for a wider class of problems.

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